

Kadec-Klee property for convergence in measure of noncommutative Orlicz spaces[☆]

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Abstract

In this paper, we study the Kadec-Klee property for convergence in measure of noncommutative Orlicz spaces $L_\varphi(\widetilde{\mathcal{M}}, \tau)$, where $\widetilde{\mathcal{M}}$ is a von Neumann algebra, and φ is an Orlicz function. We show that if $\varphi \in \Delta_2$, $L_\varphi(\widetilde{\mathcal{M}}, \tau)$ has the Kadec-Klee property in measure. As a corollary, the dual space and reflexivity of $L_\varphi(\widetilde{\mathcal{M}}, \tau)$ are given.

Keywords: Noncommutative Orlicz spaces, τ -measurable operator, von Neumann algebra, Orlicz function, Kadec-Klee property in measure

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1. Preliminaries

As is well known, the Kadec-Klee property was firstly studied by J. Radon [20]. This property said that if $(E, \|\cdot\|_E)$ is a normed linear space, then E is said to have the Kadec-Klee property (sometimes called the Radon-Riesz property, or property (H)) if and only if sequential weak convergence on the unit sphere coincides with norm convergence. For example, in [22] and [23], F. Riesz showed that the classical L_p -spaces, $1 < p < \infty$ have the Kadec-Klee property.

In this paper, we study the Kadec-Klee property for convergence in measure of noncommutative Orlicz spaces $L_\varphi(\widetilde{\mathcal{M}}, \tau)$ [7]. Namely, if for any $x \in L_\varphi(\widetilde{\mathcal{M}}, \tau)$ and any sequence (x_n) in $L_\varphi(\widetilde{\mathcal{M}}, \tau)$ such that $\|x_n\| \rightarrow \|x\|$ and $x_n \rightarrow x$ in measure, we have $\|x_n - x\| \rightarrow 0$ [16, 7].

If E is a Banach space, we define a order “ \leq ” on E , then the Banach space E is said to be the order continuous if for any element $x \in E$ and any sequence (x_n) in E_+ (the positive cone in E) with $0 \leq x_n \leq |x|$ and $x_n \rightarrow 0$ m -a.e., there holds $\|x_n\| \rightarrow 0$. We note that the norm $\|\cdot\|_E$ on the symmetric space E is order continuous if and only if E is separable.

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As usual, E is said to be lower locally uniformly monotone ($E \in (LLUM)$ for short), whenever for any $x \in E_+$ with $\|x\|_E = 1$ and any $\varepsilon \in (0, 1)$ there is $\delta = \delta(x, \varepsilon) \in (0, 1)$ such that the conditions $0 \leq y \leq x$ and $\|y\|_E \geq \varepsilon$ imply $\|x - y\| \leq 1 - \delta$.

20 E is said to be upper locally uniformly monotone ($E \in (ULUM)$ for short), whenever for any $x \in E_+$ with $\|x\|_E = 1$ and any $\varepsilon > 0$ there is $\delta = \delta(x, \varepsilon) > 0$ such that the conditions $y \geq 0$ and $\|y\|_E \geq \varepsilon$ imply $\|x + y\| \geq 1 + \delta$ [11].

It is useful to formulate the local uniform monotonicity properties sequentially. Clearly, $E \in (LLUM)$ (resp. $E \in (ULUM)$) if and only if for any $x \in E_+$, $x \neq 0$, and each sequence (x_n) in E_+ such that $x_n \leq x$ (resp. $x \leq x_n$) and $\|x_n\|_E \rightarrow \|x\|_E$, there holds $\|x_n - x\|_E \rightarrow 0$.

Now, we collect some of the basic facts and notation that will be used in this paper. Noncommutative integration theory was first introduced by Irving Segal [24], and is a fundamental tool in many theories, such as operator theory and statistical model [14]. In this paper we study some aspects of the theory of noncommutative Orlicz spaces, that is, spaces of measurable operators associated to a noncommutative Orlicz functional. The theory of Orlicz spaces associated to a trace was introduced by Muratov [17] and Kunze [13] and were respectively defined by Kunze [13] and Rashed et al [2] in an algebraic way and by Sadeghi [10] employing modular spaces. In this paper we take Sadeghi's approach and continue this line of investigation.

From now on, by \mathcal{M} we denote a semi-finite von Neumann algebra acting on a Hilbert space \mathcal{H} with a normal semi-finite faithful trace τ . The identity in \mathcal{M} is denoted by $\mathbf{1}$ and we denote by $\mathcal{P}(\mathcal{M})$ the complete lattice of all self-adjoint projections in \mathcal{M} . A densely-defined closed linear operator $x : \mathcal{D}(x) \rightarrow \mathcal{H}$ with domain $\mathcal{D}(x) \subseteq \mathcal{H}$ is called affiliated with \mathcal{M} if and only if $u^*xu = x$ for all unitary operators u belonging to the commutant \mathcal{M}' of \mathcal{M} . Clearly, if $x \in \mathcal{M}$ then x is affiliated with \mathcal{M} . If x be a (densely-defined closed) operator affiliated with \mathcal{M} and $x = u|x|$ be the polar decomposition, where $|x| = (x^*x)^{\frac{1}{2}} = \int_0^\infty \lambda de_\lambda(|x|)$ be the spectral decomposition and u is a partial isometry, then x said to be τ -measurable if and only if there exists a number $\lambda \geq 0$ such that $\tau(e_{(\lambda, \infty)}(|x|)) < \infty$, where $e_{[0, \lambda]}$ is the spectral resolution of $|x|$. The collection of all τ -measurable operators is denoted by $\widetilde{\mathcal{M}}$. We say that $\{x_n\}$ converges to x in measure topology ($x_n \xrightarrow{\tau_m} x$ for short), if $\lim_{n \rightarrow \infty} \tau(e_{(\varepsilon, \infty)}(|x_n - x|)) = 0$ for any $\varepsilon > 0$ [19].

In the setting of τ -measurable operators, the generalized singular value functions are the analogue (and actually, generalization) of the decreasing rearrangements of functions in the classical settings. In details, for $x \in \widetilde{\mathcal{M}}$, the generalized singular value function $\mu(x) : [0, \infty] \rightarrow [0, \infty]$ is defined by

$$\mu_t(x) = \inf\{s \geq 0 : \tau(e_{(s, \infty)}(|x|)) \leq t\}, \quad t > 0.$$

It is well known that the $\mu_{(\cdot)}(x)$ is a decreasing right-continuous function on the positive half-line $[0, \infty)$ [9].

If $x \in \widetilde{\mathcal{M}}$ and $x \geq 0$, then

$$\tau(x) = \int_0^\infty \mu_t(x) dt$$

and for a continuous function φ on $[0, \infty)$ with $\varphi(0) = 0$, we have [9]

$$\tau(\varphi(|x|)) = \int_0^\infty \varphi(\mu_t(x)) dt.$$

Next we recall the definition and some basic properties of noncommutative Orlicz spaces.

A convex function $\varphi : [0, \infty) \rightarrow [0, \infty]$ is called an Orlicz function if it is nondecreasing and continuous for $\alpha > 0$ and such that $\varphi(0) = 0$, $\varphi(\alpha) > 0$ and $\varphi(\alpha) \rightarrow \infty$ as $\alpha \rightarrow \infty$ [6]. Further we say an Orlicz function φ satisfies the Δ_2 -condition, shortly $\varphi \in \Delta_2$, if there exists a constant $k > 0$ such that $\varphi(2u) \leq k\varphi(u)$ for all $u > 0$. Generally speaking, Δ_2 -condition plays a very important role in the theory of either classic Orlicz spaces [6] or noncommutative classic Orlicz [10, 15]. For the background of Orlicz functions and Orlicz spaces one can see [21, 6].

Suppose $x \in \widetilde{\mathcal{M}}$ and φ is an Orlicz function, if we denote $\widetilde{\rho}_\varphi(x) = \tau(\varphi(|x|))$, then $\tau(\varphi(|x|))$ is a convex modular [10], hence we can define a corresponding modular space which is named noncommutative Orlicz space as follows:

$$L_\varphi(\widetilde{\mathcal{M}}, \tau) = \{x \in \widetilde{\mathcal{M}} : \tau(\varphi(\lambda|x|)) < \infty \text{ for some } \lambda > 0\}.$$

We equip this space with the Luxemburg norm

$$\|x\| = \inf\{\lambda > 0 : \tau\left(\varphi\left(\frac{|x|}{\lambda}\right)\right) \leq 1\}.$$

In the case when $\varphi(x) = |x|^p$, $1 \leq p < \infty$ for any τ -measurable operator $x \in \widetilde{\mathcal{M}}$, then $\varphi \in \Delta_2$ and $L_\varphi(\widetilde{\mathcal{M}}, \tau)$ is nothing but the noncommutative space $L_p(\widetilde{\mathcal{M}}, \tau) = \{x \in \widetilde{\mathcal{M}} : \tau(|x|^p) < \infty\}$ [15] and the Luxemburg norm generated by this function is expressed by the formula

$$\|x\|_p = (\tau(|x|^p))^{\frac{1}{p}}.$$

One can define another norm on $L_\varphi(\widetilde{\mathcal{M}}, \tau)$ as follows

$$\|x\|^o = \sup\{\tau(|xy|) : y \in L_\psi(\widetilde{\mathcal{M}}, \tau) \text{ and } \tau(\psi(y)) \leq 1\},$$

where $\psi : [0, \infty) \rightarrow [0, \infty]$ defined by $\psi(u) = \sup\{uv - \varphi(v) : v \geq 0\}$. Here we call ψ the complementary function of φ .

For more information on the theory of noncommutative Orlicz spaces we refer the reader to [17, 2, 3, 10, 13, 15].

2. Main results

65 In this section, we firstly prove that $L_\varphi(\widetilde{\mathcal{M}}, \tau)$ have Kadec-Klee property for convergence in measure implies $\varphi \in \Delta_2$. And, we find that $\varphi \in \Delta_2$ is necessary of this property. As a corollary of the Theorem 2.2, $L_\varphi(\widetilde{\mathcal{M}}, \tau)$ is order continuous, hence the Köthe dual is identified the Banach dual.

Theorem 2.1. *If $L_\varphi(\widetilde{\mathcal{M}}, \tau)$ has Kadec-Klee property for convergence in measure,*
70 *then $\varphi \in \Delta_2$.*

Proof. Suppose $\varphi \notin \Delta_2$, we choose $\{u_k\}_{k=1}^\infty \in \widetilde{\mathcal{M}}$ and select mutually orthogonal projections $\{e_k\}_{k=1}^\infty \in \mathcal{P}(\mathcal{M})$ in \mathcal{M} with $\tau(e_n) \rightarrow 0$ such that

$$\varphi\left(\left(1 + \frac{1}{k}u_k\right)\right) > 2^k\varphi(u_k)$$

and

$$\varphi(u_k)\tau(e_k) = \frac{1}{2^k},$$

where $k \in \mathbb{N}$.

Define $x = \sum_{k=1}^\infty u_k e_k$ and $x_n = \sum_{k=1}^\infty u_k e_k - u_n e_n$ then $\|x_n\| \rightarrow \|x\|$ and for any $s > 0$, by Lemma 2.6 of [9] and $\tau(e_n) \rightarrow 0$,

$$\begin{aligned} \tau(e_{(s,\infty)}(|x_n - x|)) &= \tau(e_{(s,\infty)}(|2u_n e_n|)) \\ &= \int_0^\infty \chi_{(s,\infty)}(\mu_t(|2u_n e_n|)) dt \\ &\rightarrow 0 \end{aligned}$$

Hence, $x_n \xrightarrow{\tau_m} x$.

75 On the other side, by (ii) of Proposition 3.4 in [10] and Remark 3.3 in [9],

$$\begin{aligned} 1 \geq \tau\left(\varphi\left(\frac{e_k}{\|e_k\|}\right)\right) &= \int_0^\infty \varphi\left(\mu_t\left(\frac{e_k}{\|e_k\|}\right)\right) dt \\ &= \varphi\left(\frac{1}{\|e_k\|}\right) \tau(e_k) \\ &= \varphi\left(\frac{1}{\|e_k\|}\right) \frac{1}{2^k \varphi(u_k)} \\ &> \varphi\left(\frac{1}{\|e_k\|}\right) \frac{1}{\varphi\left((1 + \frac{1}{k})u_k\right)} \end{aligned}$$

since $\mu_t(e_k) = \chi_{[0, \tau(e_k))}(t)$ for any $k \in \mathbb{N}$.

Then we have $\|e_k\| > \left[(1 + \frac{1}{k}) u_k\right]^{-1}$ and

$$\|x - x_n\| = \|2u_n e_n\| > 2 \left(1 + \frac{1}{n}\right)^{-1}$$

which completes the proof. \square

In order to prove that $L_\varphi(\widetilde{\mathcal{M}}, \tau)$ has the Kadec-Klee property for convergence in measure, we need the following two Lemmas.

Lemma 2.1. *If $\varphi \in \Delta_2$, then for any sequence $\{x_n\}$ in $L_\varphi(\widetilde{\mathcal{M}}, \tau)$, we have $\|x_n\| \rightarrow \|x\|$ if and only if $\tau(\varphi(x_n)) \rightarrow \tau(\varphi(x))$.*

Proof. Without loss of generality, suppose that $\|x\| = 1$.

If $\tau(\varphi(x_n)) \rightarrow 1$, since $\tau(\varphi(x)) \leq \|x\|$ if $\tau(\varphi(x)) \leq 1$ and $\|x\| \leq \tau(\varphi(x))$ if $\tau(\varphi(x)) > 1$ by Proposition 3.4 in [10]. Therefore $|\|x_n\| - 1| \leq |\tau(\varphi(x_n)) - 1|$ which implies $\|x_n\| \rightarrow 1$, since $\tau(\varphi(x_n)) \rightarrow 1$.

Now, assuming that $\|x_n\| \rightarrow 1$, we firstly need to consider two cases:

Case 1: If $\|x_n\| \uparrow 1$ and the result is not true, then suppose that there exists an $\varepsilon_0 > 0$ and $\{x_n\} \subset L_\varphi(\widetilde{\mathcal{M}}, \tau)$ such that $\tau(\varphi(|x_n|)) \leq 1 - \varepsilon_0$. Assume that $\|x_n\| \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. Set $a_n = \frac{1}{\|x_n\|} - 1$, then $a_n \leq 1$ for any $n \in \mathbb{N}$ and $a_n \downarrow 0$ as $n \rightarrow \infty$. Since $\varphi \in \Delta_2$, then $\sup_n \{\tau(\varphi(2|x_n|))\} < \infty$, from (iii) of Theorem 4.4 in [9] we have

$$\begin{aligned} 1 &= \tau \left(\varphi \left(\frac{|x_n|}{\|x_n\|} \right) \right) \\ &= \tau (\varphi (a_n |2x_n| + (1 - a_n) |x_n|)) \\ &= \int_0^\infty \varphi (\mu_t (a_n |2x_n| + (1 - a_n) |x_n|)) dt \\ &\leq \int_0^\infty \varphi (\mu_t (a_n |2x_n|) + \mu_t ((1 - a_n) |x_n|)) dt \\ &= \int_0^\infty \varphi (a_n \mu_t (2|x_n|) + (1 - a_n) \mu_t (|x_n|)) dt \\ &\leq \int_0^\infty (a_n \varphi (\mu_t (2|x_n|)) + (1 - a_n) \varphi (\mu_t (|x_n|))) dt \\ &= a_n \tau (\varphi (2|x_n|)) + (1 - a_n) \tau (\varphi (|x_n|)) \\ &\leq a_n \sup_n \{\tau (\varphi (2|x_n|))\} + (1 - a_n) (1 - \varepsilon_0) \\ &\rightarrow 1 - \varepsilon_0 < 1. \end{aligned}$$

This is a contradiction and thus finishes the proof.

Case 2: If $\|x_n\| \downarrow 1$ and the conclusion does not hold, then there exists a $\varepsilon_0 > 0$ and $\{x_n\} \subset L_\varphi(\widetilde{\mathcal{M}}, \tau)$ such that $\tau(\varphi(|x_n|)) \geq 1 + \varepsilon_0$. Assume that $\|x_n\| \leq 2$ for $n \in \mathbb{N}$. Since $\varphi \in \Delta_2$, there exists a constant $L > 0$ such that $\tau(\varphi(2|x_n|)) \leq L$ for

all $n \in \mathbb{N}$. By the assumption we have $0 \leq 1 - \frac{1}{\|x_n\|} \leq 1$ and $0 \leq 2 - \|x_n\| \leq 1$. Set $a_n = 1 - \frac{1}{\|x_n\|}$, $b_n = 2 - \|x_n\|$, then

$$0 \leq a_n + b_n = \left(1 - \frac{1}{\|x_n\|}\right) + (2 - \|x_n\|) = 3 - \left(\frac{1}{\|x_n\|} + \|x_n\|\right) \leq 1$$

for any $n \in \mathbb{N}$.

Therefore, by convexity of φ and (iii) of Theorem 4.4 in [9] we have

$$\begin{aligned} 1 + \varepsilon_0 &\leq \tau(\varphi(|x_n|)) \\ &= \tau\left(\varphi\left(a_n|2x_n| + b_n\frac{|x_n|}{\|x_n\|}\right)\right) \\ &= \int_0^\infty \varphi\left(\mu_t\left(a_n|2x_n| + b_n\frac{|x_n|}{\|x_n\|}\right)\right) dt \\ &\leq \int_0^\infty \varphi\left(\mu_t(a_n|2x_n|) + \mu_t\left(b_n\frac{|x_n|}{\|x_n\|}\right)\right) dt \\ &= \int_0^\infty \varphi\left(a_n\mu_t(|2x_n|) + b_n\mu_t\left(\frac{|x_n|}{\|x_n\|}\right)\right) dt \\ &\leq a_n \int_0^\infty \varphi(\mu_t(|2x_n|)) dt + b_n \int_0^\infty \varphi\left(\mu_t\left(\frac{|x_n|}{\|x_n\|}\right)\right) dt \\ &= a_n \tau(\varphi(|2x_n|)) + b_n \tau\left(\varphi\left(\frac{|x_n|}{\|x_n\|}\right)\right) \\ &\leq a_n L + b_n \\ &= \left(1 - \frac{1}{\|x_n\|}\right) L + (2 - \|x_n\|) \\ &\rightarrow 1, \end{aligned}$$

since $\tau\left(\varphi\left(\frac{|x_n|}{\|x_n\|}\right)\right) = 1$ for any $n \in \mathbb{N}$ and $1 - \frac{1}{\|x_n\|} \rightarrow 0$, a contradiction which finishes the proof. 95

Now, if $\|x_n\| \rightarrow 1$ and the conclusion does not hold, then there exists a subsequence $\{x_{n_j}\} \subseteq \{x_n\}$ which either $\|x_{n_j}\| \uparrow 1$ or $\|x_{n_j}\| \downarrow 1$, by Case 1 or Case 2 we can get a contradiction which can get the conclusion. □

Using Lemma 3.4 of [9], we can easily get the following Lemma,

Lemma 2.2. Suppose $\varphi \in \Delta_2$ and $x \in L_\varphi(\widetilde{\mathcal{M}}, \tau)$. For any sequence $\{x_n\}$ in $L_\varphi(\widetilde{\mathcal{M}}, \tau)$ such that $x_n \xrightarrow{\tau_m} x$, if the maps $s \rightarrow \mu_s(x)$ is continuous at $s = t$, we have that $\varphi(\mu_t(x_n)) \rightarrow \varphi(\mu_t(x))$ or $\mu_t(\varphi(x_n)) \rightarrow \mu_t(\varphi(x))$. 100

The following theorem shows that under the condition $\varphi \in \Delta_2$, $L_\varphi(\widetilde{\mathcal{M}}, \tau)$ has the Kadec-Klee property for convergence in measure.

Theorem 2.2. If $\varphi \in \Delta_2$, let x_n , ($n = 1, 2, \dots$) and x belong to $L_\varphi(\widetilde{\mathcal{M}}, \tau)$. The following two conditions are equivalent: 105

- (1) $\lim_{n \rightarrow \infty} \|x_n - x\| \rightarrow 0$,
(2) $\lim_{n \rightarrow \infty} \|x_n\| \rightarrow \|x\|$ and $x_n \xrightarrow{\tau_m} x$.

That to say $L_\varphi(\widetilde{\mathcal{M}}, \tau)$ has the Kadec-Klee property for convergence in measure
110 when $\varphi \in \Delta_2$.

Proof. (1) \Rightarrow (2): If (1) is true, first we note that $|\|x_n\| - \|x\|| \leq \|x_n - x\|$, hence $\|x_n\| \rightarrow \|x\|$.

Secondly, by (iii) of Proposition of [10], $\varphi \in \Delta_2$ implies $\|x_n - x\| \rightarrow 0 \Leftrightarrow \tau(\varphi(|x_n - x|)) = \int_0^\infty \varphi(\mu_t(x_n - x))dt \rightarrow 0$.

If $x_n \not\rightarrow x$ in measure, then for any $\varepsilon > 0$ there exists $t_0 > 0$ and $k_0 \in \mathbb{N}$ such that

$$\varphi(\mu_t(x_n - x)) = \mu_t(\varphi(x_n - x)) > \varepsilon,$$

115 for any $t \in [0, t_0)$ and any $n > k_0$. Denote $e = e_{[0, t_0)}(\varphi(|x_n - x|))$, then $\tau(e) = \int_0^\infty \chi_{[0, t_0)}(\mu_t(\varphi(|x_n - x|)))dt \leq t_0$, by (iv) (vi) of Lemma 2.5 and Lemma 4.1 in [9],

$$\begin{aligned} \tau(\varphi(|x_n - x|)) &= \int_0^\infty \mu_t(\varphi(|x_n - x|))dt \\ &\geq \sup \tau(e\varphi(|x_n - x|))e \\ &= \int_0^{t_0} \mu_t(\varphi(|x_n - x|))dt \\ &> \varepsilon t_0 \end{aligned}$$

this contradicts with (1).

(2) \Rightarrow (1): By the convexity of φ and (v),(vi) of Lemma 2.5 in [9], we have

$$\begin{aligned} 0 \leq \varphi\left(\mu_t\left(\frac{|x - x_n|}{2}\right)\right) &= \varphi\left(\frac{1}{2}\mu_t(|x - x_n|)\right) \\ &\leq \varphi\left(\frac{1}{2}(\mu_t(u|x|u^* + v|x_n|v^*))\right) \\ &= \varphi\left(\mu_t\left(\frac{u}{\sqrt{2}}|x|\frac{u^*}{\sqrt{2}} + \frac{v}{\sqrt{2}}|x_n|\frac{v^*}{\sqrt{2}}\right)\right) \\ &\leq \varphi\left(\mu_{\frac{t}{2}}\left(\frac{u}{\sqrt{2}}|x|\frac{u^*}{\sqrt{2}}\right) + \mu_{\frac{t}{2}}\left(\frac{v}{\sqrt{2}}|x_n|\frac{v^*}{\sqrt{2}}\right)\right) \\ &\leq \varphi\left(\frac{1}{2}\mu_{\frac{t}{2}}(|x|) + \frac{1}{2}\mu_{\frac{t}{2}}(|x_n|)\right) \\ &\leq \frac{1}{2}\left[\varphi\left(\mu_{\frac{t}{2}}(|x|)\right) + \varphi\left(\mu_{\frac{t}{2}}(|x_n|)\right)\right]. \end{aligned}$$

If $x_n \xrightarrow{\tau_m} x$, it follows from Lemma 3.1 of [9] that $\lim_{n \rightarrow \infty} \mu_t(x_n - x) = 0$ for each
120 $t > 0$, and by Lemma 2.1, suppose that $\tau(\varphi(|x_n|)) \rightarrow \tau(\varphi(|x|))$, the Fatou's Lemma

and Lemma 2.2 imply that

$$\begin{aligned}
0 \leq \int_0^\infty \varphi\left(\mu_{\frac{t}{2}}(|x|)\right) dt &= \int_0^\infty \lim_{n \rightarrow \infty} \left[\frac{\varphi\left(\mu_{\frac{t}{2}}(|x|)\right) + \varphi\left(\mu_{\frac{t}{2}}(|x_n|)\right)}{2} \right. \\
&\quad \left. - \varphi\left(\mu_t\left(\frac{|x - x_n|}{2}\right)\right) \right] dt \\
&\leq \underline{\lim}_{n \rightarrow \infty} \int_0^\infty \left[\frac{\varphi\left(\mu_{\frac{t}{2}}(|x|)\right) + \varphi\left(\mu_{\frac{t}{2}}(|x_n|)\right)}{2} \right. \\
&\quad \left. - \varphi\left(\mu_t\left(\frac{|x - x_n|}{2}\right)\right) \right] dt \\
&= \int_0^\infty \varphi\left(\mu_{\frac{t}{2}}(|x|)\right) dt - \overline{\lim}_{n \rightarrow \infty} \tau\left(\varphi\left(\frac{|x - x_n|}{2}\right)\right).
\end{aligned}$$

Then we obtain

$$- \overline{\lim}_{n \rightarrow \infty} \sup \tau\left(\varphi\left(\frac{|x - x_n|}{2}\right)\right) \geq 0,$$

which implies $\tau\left(\varphi\left(\frac{|x_n - x|}{2}\right)\right) \rightarrow 0$.

Hence $\|x_n - x\| \rightarrow 0$ since $\varphi \in \Delta_2$. This completes the proof. \square

As an application, using Theorem 2.2 we can get the following Corollary which
125 was firstly proved in [9].

Corollary 2.1. *Let x_n and x be element in $L_p(\widetilde{\mathcal{M}}, \tau)$ ($1 < p < \infty$). Then the following two conditions are equivalent:*

- (1) $\lim_{n \rightarrow \infty} \|x_n - x\|_p \rightarrow 0$,
- (2) $\lim_{n \rightarrow \infty} \|x_n\|_p \rightarrow \|x\|_p$ and $x_n \xrightarrow{\tau_m} x$.

130 In other words, the space $L_p(\widetilde{\mathcal{M}}, \tau)$ has the Kadec-Klee property for convergence in measure.

Combined with the Theorem 2.2 and Lemma 2.1 we can get

Corollary 2.2. *If $\varphi \in \Delta_2$, let x_n , ($n = 1, 2, \dots$) and x belong to $L_\varphi(\widetilde{\mathcal{M}}, \tau)$ with $x_n \xrightarrow{\tau_m} x$, then*

- 135 (1) *The noncommutative Orlicz spaces $L_\varphi(\widetilde{\mathcal{M}}, \tau)$ has the property LLUM.*
- (2) *The noncommutative Orlicz spaces $L_\varphi(\widetilde{\mathcal{M}}, \tau)$ has the property ULUM.*

From Lemma 3.1 of [9] and Lemma 2.1 we have

Theorem 2.3. *Suppose that $\varphi \in \Delta_2$. The noncommutative Orlicz spaces $L_\varphi(\widetilde{\mathcal{M}}, \tau)$ is order continuous. Hence, it is separable. Especially, $L_p(\widetilde{\mathcal{M}}, \tau)$ is separable, where
140 $1 < p < \infty$.*

Next, we consider the dual space of $L^\varphi(\widetilde{\mathcal{M}}, \tau)$. By $L^0(\mathbb{R}^+, m)$ we denote the space of all \mathbb{C} -valued Lebesgue measurable function of \mathbb{R}^+ . A Banach space $(E, \|\cdot\|_E)$, where $E \subseteq L^0(\mathbb{R}^+, m)$, is called the rearrangement-invariant Banach function space if it follows from $f \in E, g \in L^0(\mathbb{R}^+, m)$ and $\mu(g) \leq \mu(f)$ that $g \in E$ and $\|g\|_E \leq \|f\|_E$. Furthermore, $(E, \|\cdot\|_E)$ is called a symmetric Banach function space if it has the additional property, that $f, g \in E$ and $g \prec\prec f$ imply that $\|g\|_E \leq \|f\|_E$. Here $g \prec\prec f$ denotes for all $t > 0$:

$$\int_0^t \mu_s(g) ds \leq \int_0^t \mu_s(f) ds.$$

If the Banach space $E \subseteq \widetilde{\mathcal{M}}$ is properly symmetric then the Köthe dual E^\times is defined by setting

$$E^\times = \{y \in \widetilde{\mathcal{M}} : xy \in L_1(\mathcal{M}) \text{ for all } x \in E\}$$

and if $x \in \widetilde{\mathcal{M}}$, we define

$$\|x\|_{E^\times} = \sup\{\tau(|xy|) : y \in E, \|y\|_E \leq 1\}.$$

Next theorem shows the dual space of the $L_\varphi(\widetilde{\mathcal{M}}, \tau)$.

Theorem 2.4. *If $\varphi \in \Delta_2$, we have*

$$L_\varphi(\widetilde{\mathcal{M}}, \tau)^* = L_\psi^o(\widetilde{\mathcal{M}}, \tau),$$

where $L_\psi^o(\widetilde{\mathcal{M}}, \tau) = (L_\psi(\widetilde{\mathcal{M}}, \tau), \|\cdot\|^o)$.

Proof. Theorem 5.11 combined with Theorem 5.6 of [8] show that for a rearrangement invariant symmetric Banach function space $E(\mathcal{M})$, if it is order continuous, then
145 Banach dual $E(\mathcal{M})^*$ may be identified with the space $E^\times(\mathcal{M})$ if $E(\mathcal{M})$. Hence, by Theorem 2.3, we can get the conclusion. \square

Similar to the classic case, using Theorem 2.4 we can get

Corollary 2.3. *$L_\varphi(\widetilde{\mathcal{M}}, \tau)$ is reflexive if and only if both $\varphi \in \Delta_2$ and $\psi \in \Delta_2$.*

It easy to know that if $\varphi(x) = |x|^p$ ($1 < p < \infty$), then $\psi(x) = |x|^q$ is complementary function of φ , where $\frac{1}{p} + \frac{1}{q} = 1$. Hence, as an special example of Theorem 2.4,
150 one have

(1) $L_p(\widetilde{\mathcal{M}}, \tau)^* = L_q(\widetilde{\mathcal{M}}, \tau)$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$;

(2) $L_p(\widetilde{\mathcal{M}}, \tau)$ is reflexive when $1 < p < \infty$.

Especially, if $p = 1$ then $L_1(\widetilde{\mathcal{M}}, \tau)^* = L_\infty(\widetilde{\mathcal{M}}, \tau) = \mathcal{M}$, but $L_1(\widetilde{\mathcal{M}}, \tau)$ is nonre-
155 flexive since $\mathcal{M}^* \neq L_1(\widetilde{\mathcal{M}}, \tau)$.

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